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Short communication

Numerical evaluation of high-order modes of vibration in uniform Euler–Bernoulli beams

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Abstract

Many vibration text books give expressions for the mode shape functions of uniform Euler–Bernoulli beams. However, the common forms of these expressions permit the evaluation of only the first 12 modes or so due to numerical issues. This article presents alternative and approximate forms for the evaluation of beam mode shape functions that are numerically stable. Although the approximations allow numerical evaluation of the mode shapes for all modes of vibration, the penalty is that some errors occur in the calculations for low-order modes, and these errors are quantified. Beams with combinations of the classical boundary conditions of clamped, free, pinned, and sliding are covered.

1. Introduction

The numerical evaluation of mode shape functions is important when modeling a structure subject to forced vibration, as the response can be represented as a summation of modes. Many vibration text books give expressions for the mode shape functions of uniform Euler–Bernoulli beams, for example Refs. [1–3]. However, the common forms of these expressions permit the evaluation of only the first 12 modes or so due to numerical issues. Shankar and Keane [4] discussed this problem when they calculated the individual modal responses of beams that make up a large truss structure. They reformulated the expression for the mode shape of a free–free beam so that it could be numerically evaluated up to the first two hundredth mode or so. Warburton [5], also presents mode shape expressions for transverse vibration in plates based on beam mode-shape functions, which include some boundary conditions that are numerical stable up to a high number of modes, but eventually become unstable for a very high number of modes. Tang [6] has also discussed the problem of numerically evaluating high-order beam mode shapes, and gave expressions for beam mode shape functions with some other boundary conditions. However, these expressions, and others found in the literature cannot be used to numerically evaluate very-high-order beam mode shapes.

This article discusses the reasons for the numerical problems and presents alternative, compact mode shape expressions for beams with a combination of clamped, free, pinned, and sliding boundary conditions based on the method described by Shankar and Keane [4]. Although these expressions are useful for up to the first two

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hundred modes or so, they also suffer from numerical problems for higher modes. To overcome these problems, approximate expressions for the mode shape functions are derived. These allow numerical evaluation of the mode shapes for all modes of beam vibration. However, the penalty is that some errors occur in the calculations for low-order modes, and these errors are quantified. The expressions for the mode shapes given in this article are useful for slender beams where Euler–Bernoulli theory applies, which is when the wavelength of a flexural wave is greater than about six times the thickness of the beam [7]. In terms of the nth mode of vibration the criterion is approximately $l/t > n\pi$ where l and t are the length and thickness of the beam, respectively. The mode shape expressions can also be used for the mode shapes of thin plates which are approximated by Euler–Bernoulli beam mode shapes.

2. Beam vibration

The equation of motion of free vibration of a uniform Euler–Bernoulli beam is given by

$$EI\frac{\partial^4 w(x)}{\partial x^4} + \rho S\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad (1)$$

where E, I, ρ, and S are the Young's modulus, second moment of area, density, and cross-sectional area, respectively. Assuming time harmonic motion at angular frequency ω of the form $e^{j\omega t}$, but neglecting it in the subsequent expressions for clarity, the displacement of the beam can be written in terms of evanescent and propagating waves as

$$W(x) = A_1 e^{kx} + A_2 e^{-kx} + A_3 e^{jkx} + A_4 e^{-jkx}, \quad (2)$$

where A_1 is an evanescent wave decaying to the left, A_2 is an evanescent wave decaying to the right, A_3 is a wave propagating to the left and A_4 is a wave propagating to the right; k is the flexural wavenumber and is given by $k = (\rho S/EI)^{1/4}\omega^{1/2}$. By applying the boundary conditions, the wave amplitudes A_1–A_4 relative to one of the wave amplitudes can be calculated at a natural frequency, and hence the mode shape function determined. For beams with boundary conditions other than pinned, the mode shape function will include evanescent waves and it is these that cause a problem with the numerical evaluation of higher-order mode shapes.

It is more usual in text books for the displacement to be written as

$$W(x) = A\sin kx + B\cos kx + C\sinh kx + D\cosh kx, \quad (3)$$

where the trigonometric functions represent the propagating waves, and the hyperbolic functions represent the evanescent waves. As an example, the mode shape function of a clamped–clamped beam is given by

$$\phi_n(x) = (\cosh k_n x - \cos k_n x) - \sigma_n(\sinh k_n x - \sin k_n x), \quad (4)$$

where

$$\sigma_n = \frac{\cosh k_n l - \cos k_n l}{\sinh k_n l - \sin k_n l}, \quad (5a)$$

in which the subscript n denotes the nth mode of vibration. The problem with this formulation is that when $k_n l$ is large the hyperbolic functions become very large; the trigonometric functions are, of course, less than or equal to unity. Eq. (5a), for large n becomes:

$$\sigma_n \approx \frac{\cosh k_n l}{\sinh k_n l} = \frac{e^{k_n l} + e^{-k_n l}}{e^{k_n l} - e^{-k_n l}} \approx 1 \quad (5b)$$

and Eq. (4) reduces to

$$\phi_n(x) \approx (\cosh k_n x - \cos k_n x) - (\sinh k_n x - \sin k_n x). \quad (5c)$$

When $k_n l \gg 1$ and $x \approx l$, this becomes:

$$\phi_n(l) \approx \cosh k_n l - \sinh k_n l, \quad (6)$$

which means that a small number is obtained by subtracting two very large numbers. This leads to gross inaccuracy in the mode shape for positions on the beam close to where $x = l$. The problem can be overcome by rearranging the equations so that subtraction of large, near equal terms does not occur. This is again illustrated using a clamped–clamped beam. Eq. (5a) can be rewritten as

$$\sigma_n = [1 + \upsilon], \tag{7}$$

where

$$\upsilon = \frac{e^{-k_n l} - \cos k_n l + \sin k_n l}{\sinh k_n l - \sin k_n l},$$

which tends to zero when $k_n l \gg 1$. Thus Eq. (4) can be written as

$$\phi_n(x) = e^{-k_n x} - \cos k_n x + [1 + \upsilon] \sin k_n x - \upsilon \sinh k_n x. \tag{8}$$

Eq. (8) is an *exact* solution for the nth mode shape. This form is more accurate than Eq. (4) for high n; however, it is still numerically ill-conditioned for very high n (greater than about 200 using double precision floating point arithmetic with MATLAB) because of the term $\upsilon \sinh k_n x$. The variable υ tends to zero as k_n increases and $\sinh k_n l$ becomes very large so their product yields a finite quantity. The numerical ill-conditioning occurs when the argument of the hyperbolic functions leads to a quantity larger than that which can be represented by the machine (in most programming languages this number is about 1.79×10^{308}), Note that $\cosh 710 \approx \sinh 710 \approx 1.117 \times 10^{308}$.

By making an approximation, Eq. (8) can be modified so that it is numerically well-conditioned for $n \to \infty$ (providing, of course, that Euler–Bernoulli beam theory is still valid for the beam under consideration). First note that $\upsilon \sinh k_n x$ can be written as

$$\upsilon \sinh k_n x = \left(\frac{e^{-k_n l} - \cos k_n l + \sin k_n l}{\sinh k_n l - \sin k_n l} \right) \sinh k_n x. \tag{9}$$

For large k_n, $\sin k_n l$ can be neglected compared to $\sinh k_n l$, in the denominator of Eq. (9). By also making the approximation $\sinh k_n x / \sinh k_n l \approx e^{k_n(x-l)}$, equations (8) and (9) combine to give

$$\begin{aligned} \phi_n(x) = {} & e^{-k_n x} - \cos k_n x + [1 + \upsilon] \sin k_n x \\ & - [e^{-k_n l} - \cos k_n l + \sin k_n l] e^{k_n (x-l)}. \end{aligned} \tag{10}$$

The approximation introduces an error in the mode shape calculation for the first few modes. However, this error tends to zero as k_n increases. The mode shape functions are normalised such that $\int_0^l \phi_n^2(x)\,dx = l$, thus to quantify the error the following expression is used:

$$e = \left| \frac{\int_0^l \phi_n^2(x)\,dx - \int_0^l \phi_n^2(x)_{\text{approx.}}\,dx}{\int_0^l \phi_n^2(x)\,dx} \right|, \tag{11}$$

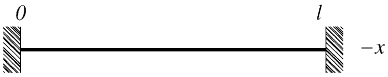
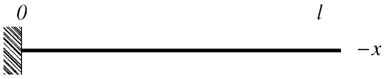
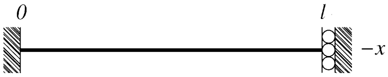
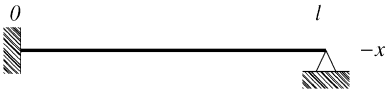
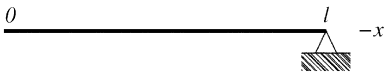
which simplifies to $e = |l - l^*/l|$ where l^* is a length that is slightly different to the actual length of the beam. The error for the first six modes using Eq. (10) is given in Table 1. It can be seen for $n > 5$, the approximation for the mode shape has negligible error.

Similar expressions for mode shape functions can be calculated for beams with other boundary conditions in a straightforward manner, and these are given in Table 2 for combinations of the classical boundary

Table 1
Error in using approximate mode shape function for a clamped–clamped beam

n	$e = \lvert l - l^*/l \rvert$
1	$O(10^{-2})$
2	$O(10^{-3})$
3	$O(10^{-5})$
4	$O(10^{-6})$
5	$O(10^{-8})$
6	$< O(10^{-9})$

Table 2
Natural frequencies and mode shape functions for a uniform Euler–Bernoulli beam with various boundary conditions

Boundary condition		Equations	n	$k_n l$	Order of error
	(1)	$\phi(0)=\phi(l)=\phi'(0)=\phi'(l)=0$	1	4.73004	10^{-2}
	(2)	$\cos k_n l \cosh k_n l = 1$	2	7.85320	10^{-3}
	(3)	$\phi(x)=a_1-b_1+[1+\upsilon_1]b_2-\upsilon_1 c_2$	3	10.9956	10^{-5}
	(4)	$\phi(x)=a_1-b_1+[1+\upsilon_1]b_2-[a_3-b_3+b_4]d_1$	4	14.1372	10^{-6}
			5	17.2788	10^{-8}
			6, 7, …	$(2n+1)\pi/2$	$\leqslant 10^{-9}$
	(1)	$\phi(0)=\phi''(l)=\phi'(0)=\phi'''(l)=0$	1	1.87510	10^{-2}
	(2)	$\cos k_n l \cosh k_n l = -1$	2	4.69409	10^{-3}
	(3)	$\phi(x)=a_1-b_1+[1+\upsilon_1]b_2-\upsilon_2 c_2$	3	7.85476	10^{-4}
	(4)	$\phi(x)=a_1-b_1+[1+\upsilon_1]b_2-[a_3-b_3+b_4]d_1$	4	10.9955	10^{-5}
			5	14.1372	10^{-8}
			6, 7, …	$(2n-1)\pi/2$	$\leqslant 10^{-9}$
	(1)	$\phi(0)=\phi'(l)=\phi'(0)=\phi'''(l)=0$	1	2.36502	10^{0}
	(2)	$\tan k_n l + \tanh k_n l = 0$	2	5.49780	10^{-2}
	(3)	$\phi(x)=a_1-b_1+[1+\upsilon_4]b_2-\upsilon_4 c_2$	3	8.63938	10^{-4}
	(4)	$\phi(x)=a_1-b_1+[1+\upsilon_4]b_2-d_2$	4	11.7810	10^{-5}
			5	14.9226	10^{-7}
			6, 7, …	$(4n-1)\pi/4$	$\leqslant 10^{-8}$
	(1)	$\phi''(0)=\phi''(l)=\phi'''(0)=\phi'''(l)=0$	1	3.92660	10^{-4}
	(2)	$\tan k_n l - \tanh k_n l = 0$	2	7.06858	10^{-6}
	(3)	$\phi(x)=a_1-b_1+[1+\upsilon_4]b_2-\upsilon_4 c_2$	3	10.2102	10^{-9}
	(4)	$\phi(x)=a_1-b_1+[1+\upsilon_4]b_2-d_2$	4	13.3518	10^{-12}
			5	16.4934	$\leqslant 10^{-15}$
			6, 7, …	$(4n+1)\pi/4$	$\leqslant 10^{-15}$
	(1)	$\phi''(0)=\phi''(l)=\phi'''(0)=\phi'''(l)=0$	1	4.73004	10^{-2}
	(2)	$\cos k_n l \cosh k_n l = 1$	2	7.85320	10^{-3}
	(3)	$\phi(x)=a_1+b_1-[1+\upsilon_1]b_2-\upsilon_1 c_2$	3	10.9956	10^{-5}
	(4)	$\phi(x)=a_1+b_1-[1+\upsilon_1]b_2-[a_3-b_3+b_4]d_1$	4	14.1372	10^{-6}
			5	17.2788	10^{-8}
			6, 7, …	$(2n+1)\pi/2$	$\leqslant 10^{-9}$
	(1)	$\phi''(0)=\phi'(l)=\phi'''(0)=\phi'''(l)=0$	1	2.36502	10^{-1}
	(2)	$\tan k_n l + \tanh k_n l = 0$	2	5.49780	10^{-2}
	(3)	$\phi(x)=a_1+b_1-[1+\upsilon_4]b_2-\upsilon_4 c_2$	3	8.63938	10^{-4}
	(4)	$\phi(x)=a_1+b_1-[1+\upsilon_4]b_2-d_2$	4	11.7810	10^{-5}
			5	14.9226	10^{-7}
			6, 7, …	$(4n-1)\pi/4$	$\leqslant 10^{-8}$
	(1)	$\phi(0)=\phi(l)=\phi'(0)=\phi''(l)=0$	1	3.92660	10^{-6}
	(2)	$\tan k_n l - \tanh k_n l = 0$	2	7.06858	10^{-9}
	(3)	$\phi(x)=a_1+b_1-[1+\upsilon_3]b_2-\upsilon_3 c_2$	3	10.2102	10^{-12}
	(4)	$\phi(x)=a_1+b_1-[1+\upsilon_3]b_2-d_2$	4	13.3518	10^{-15}
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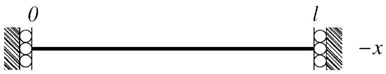
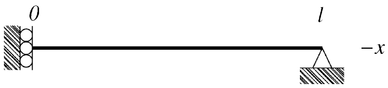
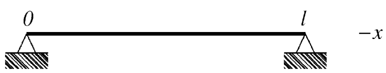
			6, 7, …	$(4n+1)\pi/4$	$\leqslant 10^{-15}$
	(1)	$\phi'(0)=\phi'(l)=\phi'''(0)=\phi'''(l)=0$			
	(2)	$\sin k_n l \sinh k_n l = 0$	1, 2, …	$n\pi$	—
	(3)	$\phi(x)=\sqrt{2}b_1$			
	(1)	$\phi'(0)=\phi(l)=\phi'''(0)=\phi''(l)=0$			
	(2)	$\cos k_n l \cosh k_n l = 0$	1, 2, …	$(2n-1)\pi/2$	—
	(3)	$\phi(x)=\sqrt{2}b_1$			
	(1)	$\phi(0)=\phi(l)=\phi''(0)=\phi''(l)=0$			
	(2)	$\sin k_n l \sinh k_n l = 0$	1, 2, …	$n\pi$	—
	(3)	$\phi(x)=\sqrt{2}b_2$			

Table 2 *(continued)*

(1) Boundary conditions; (2) characteristic or frequency equation; (3) mode shape (exact); (4) mode shape (approximation).
The variables in Table 2 are given by

$$a_1 = \mathrm{e}^{-k_n x},\ a_2 = \mathrm{e}^{k_n x},\ a_3 = \mathrm{e}^{-k_n l},\ a_4 = \mathrm{e}^{k_n l},$$

$$b_1 = \cos k_n x,\ b_2 = \sin k_n x,\ b_3 = \cos k_n l, b_4 = \sin k_n l,$$

$$c_1 = \cosh k_n x,\ c_2 = \sinh k_n x,\ c_3 = \cosh k_n l,\ c_4 = \sinh k_n l,$$

$$d_1 = \mathrm{e}^{k_n(x-l)},\ d_2 = \mathrm{e}^{k_n(x-2l)},$$

$$\upsilon_1 = \frac{a_3 - b_3 + b_4}{c_4 - b_4},\ \upsilon_2 = \frac{-a_3 - b_3 - b_4}{c_3 + b_3},\ \upsilon_3 = \frac{a_3}{c_4},\ \upsilon_4 = \frac{a_3}{c_3}.$$

conditions. For completeness, the values of $k_n l$ corresponding to the natural frequencies are also given. Similar tables giving solutions to the natural frequency equations can be found in text books, for example Refs. [3,8,9]. The last column in the table is the error in the approximate mode shape functions calculated using Eq. (11).

3. Concluding remarks

In this article, the problem of evaluating beam mode shape functions at high frequencies has been discussed. Alternative expressions that do not suffer from numerical ill-conditioning up to about 200 modes have been presented and approximate expressions valid up to an infinitely high frequency (provided that Euler–Bernoulli beam theory applies) have been derived. The results have been tabulated for ease of reference.

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